

AN INFINITESIMAL p -ADIC MULTIPLICATIVE MANIN-MUMFORD CONJECTURE

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ABSTRACT. Our results concern analytic functions on the open unit poly-disc in \mathbb{C}_p^n centered at the multiplicative unit and we prove that such functions only vanish at finitely many n -tuples of roots of unity $(\zeta_1 - 1, \dots, \zeta_n - 1)$ unless they vanish along a translate of the formal multiplicative group. For polynomial functions, this follows from the multiplicative Manin-Mumford conjecture. However we allow for a much wider class of analytic functions; in particular we establish a rigidity result for formal tori. Moreover, our methods apply to Lubin-Tate formal groups beyond just formal \mathbb{G}_m and we extend the results to this setting.

1. INTRODUCTION

The classical Manin-Mumford Conjecture, proven by M. Raynaud [Ray83], states that for an algebraic curve C of genus greater than one defined over a number field K together with an embedding defined over K of C into its Jacobian, there are only finitely many torsion points of the Jacobian on the curve, i.e. $C(\overline{K}) \cap \text{Jac}(C)(\overline{K})_{\text{tor}}$ is a finite set. One may ask a similar question replacing the Jacobian by an abelian variety, or more generally a commutative group variety G . The case of $G = \mathbb{G}_m^n$, the so-called multiplicative Manin-Mumford Conjecture, was already proven by S. Lang in [Lan60, Lan65]. It states that if an irreducible curve C embedded in \mathbb{G}_m^n contains infinitely many torsion points, it must be a translate of \mathbb{G}_m by a torsion point. Considering the case of $n = 2$ for ease of exposition, this amounts to the following explicit statement on polynomials:

Proposition 1.1 ([Lan60], p.28). *Let C be an absolutely irreducible plane curve given by the zero set of a polynomial $f(X, Y) = 0$. Assume C passes through the multiplicative origin and*

$$f(\zeta, \xi) = 0$$

for infinitely many pairs of roots of unity (ζ, ξ) . Then $f(X, Y) = X^m - Y^l$ or $f(X, Y) = X^m Y^l - 1$ for a pair of nonnegative integers $(m, l) \neq (0, 0)$.

The proof relies on the algebraic properties of the polynomial f . However, in the p -adic setting, we obtain the following statement for power series as a special case of our results:

Proposition 1.2. *Let \mathcal{O}_F denote the ring of integers of a finite extension F/\mathbb{Q}_p and $\phi \in \mathcal{O}_F[[X, Y]]$ an irreducible power series passing through the origin. Then if*

$$\phi(\zeta - 1, \xi - 1) = 0$$

for infinitely many pairs of p -power roots of unity (ζ, ξ) , after possibly switching X and Y there is $m \in \mathbb{Z}_p$ so that $\phi = (X + 1)^m - (Y + 1)$, where $(X + 1)^m = 1 + \sum_{i=1}^{\infty} \frac{m \cdots (m-i+1)}{i!} X^i$.

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Denoting $\mathcal{S} = \{\zeta - 1 \mid \zeta \in \mu_{p^\infty}(\overline{\mathbb{Q}_p})\}$, we consider more generally the set of n -tuples \mathcal{S}^n . We prove the following and deduce Proposition 1.2 when $n = 2$ and $I = (\phi)$:

Theorem 1.3. *Let $A = \mathcal{O}_F[[X_1, \dots, X_n]]/I$, where \mathcal{O}_F denotes the ring of integers of a finite extension of \mathbb{Q}_p . Then exactly one of the following occurs:*

- (1) *The formal scheme $\mathrm{Spf}(A)$ contains a translate of $\widehat{\mathbb{G}}_m$ by a torsion point of $\widehat{\mathbb{G}}_m^n$.*
- (2) *There exist an explicit constant $C_I > 0$ depending only on I and the choice of p -adic absolute value $|\cdot|_p$, as well as a finite set $\mathcal{F} \subset \mathcal{S}^n$ such that for any $\phi \in I$*

$$|\phi(\zeta_1 - 1, \dots, \zeta_n - 1)|_p > C_\phi$$

$$\text{if } (\zeta_1 - 1, \dots, \zeta_n - 1) \in \mathcal{S}^n \setminus \mathcal{F}.$$

Power series $\phi \in \mathcal{O}_F[[X_1, \dots, X_n]]$ give rise to analytic functions on the open p -adic unit polydisk $\mathbb{B}^n(e, \mathbb{C}_p)$ centered at $e = (1, \dots, 1)$ by evaluating points $Q \in \mathbb{B}^n(e, \mathbb{C}_p)$ at $Q - e$. The only roots of unity on the open disk have p -power order. The proof of Theorem 1.3, given in Section 2, relies on the fact that any infinite sequence in \mathcal{S}^n must approach the boundary of the polydisk since the normalized valuation is $v_p(\zeta_{p^k} - 1) = 1/(p^k - p^{k-1})$ when ζ_{p^k} has exact order p^k . We also utilize the action of automorphisms of the formal group $\widehat{\mathbb{G}}_m^n$ on the torsion points in \mathcal{S}^n together with the algebraic properties of formal power series rings. In fact, our methods can be used to prove that Theorem 1.3 holds in greater generality replacing $\widehat{\mathbb{G}}_m$ by a Lubin-Tate formal group \mathcal{F}_{LT} and replacing \mathcal{S}^n by n -tuples of torsion points of \mathcal{F}_{LT} . This is shown in Section 3.

Many generalizations of the Manin-Mumford Conjecture are known by work of S. Zhang [Zha95, Zha98], E. Ullmo [Ull98] and others, however to our knowledge these infinitesimal p -adic strengthenings have not previously been considered. Rather, they are related to rigidity results appearing in H. Hida's work which we discuss in Section 2.1. We remark that Theorem 1.3 makes the additional claim that there cannot be infinitely many torsion points arbitrarily close to $\mathrm{Spf}(A)(\mathbb{C}_p) \subset \mathbb{B}^n(0, \mathbb{C}_p)$ unless for a specific geometric reason. This is a purely p -adic phenomenon, for instance torsion points on abelian varieties are dense even in the complex analytic topology. It was observed by J. Tate and F. Voloch [TV96] that given a linear form f vanishing at $Z(f)$ and a choice of p -adic absolute value $|\cdot|_p$, there is a uniform bound ϵ_f such that for any n -tuple of roots of unity,

$$f(\zeta_1, \dots, \zeta_n) \neq 0 \Rightarrow |(\zeta_1, \dots, \zeta_n) - P|_p > \epsilon_f \quad \forall P \in Z(f).$$

They formulated a general conjecture for algebraic varieties which was proven by T. Scanlon [Sca98]. Our results exhibit the same phenomenon for formal power series and p -power roots of unity, whereas in her thesis A. Neira [Nei02] proves this for analytic functions on the closed disk. In this case one need not restrict to p -power roots of unity. Finally, P. Monsky [Mon81] studied such p -adic power series rings with applications to Iwasawa theory in mind and [Mon81, Section 2] established some of the results used in this paper.

The rings of analytic functions we consider are related to many interesting arithmetic objects. They occur naturally as completed group rings such as $\mathbb{Z}_p[[\mathbb{Z}_p^n]] \cong \mathbb{Z}_p[[X_1, \dots, X_n]]$, Iwasawa theory studies the ideals cut out by p -adic L -functions inside these rings, and they are completed local rings at smooth points of schemes over \mathcal{O}_F . Our initial interest in the problem was motivated by the study of families of p -adic automorphic forms parametrized by weight. The spaces of p -adic weights are up to connected components formal power series rings and one is led to consider a slightly larger class of special points corresponding to weights

of classical automorphic forms. The applications to this setting constitute the content of a subsequent paper.

2. MANIN-MUMFORD FOR FORMAL \mathbb{G}_m

Throughout this paper, we fix a prime p . We denote by \mathcal{O}_F the ring of integers of a finite extension F of the p -adic numbers \mathbb{Q}_p and denote by v_p the valuation on the p -adic complex numbers \mathbb{C}_p , normalized so that $v_p(p) = 1$.

2.1. Formal schemes and rigidity results. We review some relevant rigidity theorems found in the literature. Over a field k of characteristic p , C-L. Chai [Cha08] proves a rigidity result for p -divisible formal groups, which is used in his work together with F. Oort on the Hecke Orbit Conjecture for Siegel modular varieties (see e.g. [Cha05]). Considering the torus

$$\widehat{\mathbb{G}}_{m/k}^n = \mathrm{Spf}(k[[X_1, \dots, X_n]]),$$

let $X_*(\widehat{\mathbb{G}}_m^n) = \mathrm{Hom}_k(\widehat{\mathbb{G}}_m, \widehat{\mathbb{G}}_m^n) \cong \mathbb{Z}_p^n$ denote the group of cocharacters, so that $\mathrm{GL}(X_*) \cong \mathrm{GL}_n(\mathbb{Z}_p^n)$ naturally acts on the torus. One has from [Cha08, Theorem 4.3]:

Theorem 2.1 (Chai). *Let $k = \overline{\mathbb{F}}_p$ and $Z \subset \widehat{\mathbb{G}}_{m/k}^n$ a closed formal subscheme, equidimensional of dimension r . If Z is stable under the diagonal action for all $u \in U$ in an open subgroup of $(\mathbb{Z}_p^\times)^n \subset \mathrm{GL}(X_*)$, then there are finitely many \mathbb{Z}_p -direct summands T_1, \dots, T_s of rank r of $X_*(\widehat{\mathbb{G}}_m^n)$ so that*

$$Z = \bigcup_{i=1}^s \widehat{\mathbb{G}}_{m/k} \otimes T_i.$$

Hida uses Chai's rigidity results [Hid10, Section 3.4] and establishes characteristic zero versions thereof in [Hid11, Lemma 1.2] and [Hid14, Section 4]. He proves:

Lemma 2.2 ([Hid14], Lemma 4.1). *Let $Z = \mathrm{Spf}(\mathcal{O}_F[[X_1, \dots, X_n]]/I)$ be a closed formal subscheme of $\widehat{\mathbb{G}}_m^n$ that is flat and geometrically irreducible. Suppose there is an open subgroup $U \subseteq \mathbb{Z}_p^\times$ such that Z is stable under the action $(1 + X_i) \mapsto (1 + X_i)^u$ for all $u \in U$. If there exists a subset $\Omega \subseteq Z(\mathbb{C}_p) \cap \mu_{p^\infty}^n(\mathbb{C}_p)$ Zariski dense in Z , then Z is the translate of a formal subtorus by a torsion point in Ω .*

In particular, he obtains a rigidity result for formal power series by applying Lemma 2.2 to their graph in [Hid14, Corollary 4.2]:

Corollary 2.3 (Hida). *Let $\phi \in \mathcal{O}_F[[X_1, \dots, X_n]]$ be a power series such that there is a Zariski-dense subset $\Omega \subset \mu_{p^\infty}^n(\mathbb{C}_p)$ in $\widehat{\mathbb{G}}_m^n(\mathbb{C}_p)$ with $\phi(\zeta - 1) \subseteq \mu_{p^\infty}(\mathbb{C}_p)$ for all $\zeta \in \Omega$. Then there exist $\zeta_0 \in \mu_{p^\infty}(\mathcal{O}_F)$ and $N = (N_1, \dots, N_n) \in \mathbb{Z}_p^n$ such that $\phi(X_1, \dots, X_n) = \zeta_0 \prod_{i=1}^n (1 + X_i)^{N_i}$.*

Writing $A = \mathcal{O}_F[[X_1, \dots, X_n]]/I$ and assuming $\mathrm{Spf}(A)$ is geometrically irreducible, we therefore consider the following statements:

- (I) The formal subscheme $\mathrm{Spf}(A) \subset \widehat{\mathbb{G}}_m^n$ is the translate of a formal subtorus by a torsion point.
- (II) There is a Zariski-dense set of torsion points of $\widehat{\mathbb{G}}_m^n$ on $\mathrm{Spf}(A)(\overline{\mathbb{Q}}_p)$.
- (III) The formal subscheme $\mathrm{Spf}(A) \subset \widehat{\mathbb{G}}_m^n$ is stable under the action of an open subgroup U of the diagonal in the cocharacters $\mathrm{GL}(X_*) \cong \mathrm{GL}_n(\mathbb{Z}_p)$.

The first statement implies the two others. Chai's result is a version of (III) \Rightarrow (I) in characteristic p . In characteristic zero, Lemma 2.2 shows that (II & III) \Rightarrow (I), whereas Corollary 2.3 shows in some special cases that (II) \Rightarrow (I). In Proposition 2.9 we prove a vanishing result for arbitrary formal power series and deduce in all generality that (II) \Rightarrow (I), see in particular Corollary 2.16. To this end, we adopt the point of view of “unlikely intersection” results such as the Manin-Mumford and Andr -Oort Conjectures.

2.2. Manin-Mumford formulation. We use terminology inspired by the following formulation in [Ull07, Section 3.1.] of the classical Manin-Mumford Conjecture:

Let X/\mathbb{C} be an algebraic variety. Define a set of *special subvarieties* S_X to be the following irreducible subvarieties of X :

- If X is an abelian variety, the special subvarieties are the translates by torsion points of abelian subvarieties of X .
- If X is a torus, the special subvarieties are given by the products of torsion points with subtori.

A *special point* is a zero-dimensional special subvariety. The conjecture may then simply be stated as:

Conjecture (Manin-Mumford). *An irreducible component of the Zariski closure of a set of special points is a special subvariety.*

We are interested in the $\overline{\mathbb{Q}_p}$ -points on $\mathrm{Spec}(\mathcal{O}_F[[X_1, \dots, X_n]])$ with coordinates:

$$\mathcal{S}^n = \{(\zeta_1 - 1, \dots, \zeta_n - 1) \mid \zeta_i \in \mu_{p^\infty}(\overline{\mathbb{Q}_p})\}$$

which are precisely the torsion points of the formal Lie group $\widehat{\mathbb{G}_m^n}$. For $A = \mathcal{O}_F[[X_1, \dots, X_n]]/I$ we define the set of *special points* $\mathcal{S}_A \subset \mathrm{Spec}(A)$ to be the points of \mathcal{S}^n lying on $\mathrm{Spec}(A)$, i.e. prime ideals $\mathfrak{p} = (X_1 + 1 - \zeta_1, \dots, X_n + 1 - \zeta_n)$ containing I .

We want $\mathrm{Spec}(A)$ to be a *special subscheme* exactly when $\mathrm{Spf}(A) \subset \widehat{\mathbb{G}_m^n}$ is the product of a formal subtorus by a torsion point of $\widehat{\mathbb{G}_m^n}$. Note that endomorphisms of the formal group law act on a choice of coordinates via

$$(X_1, \dots, X_n) \mapsto \left(\prod_{j=1}^n (X_j + 1)^{a_{1,j}} - 1, \dots, \prod_{j=1}^n (X_j + 1)^{a_{n,j}} - 1 \right)$$

for matrices $(a_{i,j}) \in M_n(\mathbb{Z}_p) \cong \mathrm{End}(\widehat{\mathbb{G}_m^n})$.

For any choice of coordinates, the set of special subschemes should account for twists by automorphisms in $\mathrm{GL}_n(\mathbb{Z}_p)$. We therefore make the following definitions:

Definition 2.4.

- (1) A *multiplicative change of variables* on $\mathcal{O}_F[[X_1, \dots, X_n]]$ is given by possibly swapping the roles of X_i and X_j and a series of transformations of the form:

$$X_i \mapsto (1 + X_i) \prod_{1 \leq j < i} (1 + X_j)^{B_j} - 1$$

for $1 \leq i \leq n$, where $B_j \in \mathbb{Z}_p$.

- (2) A *special (multiplicative) subscheme* of $\mathrm{Spec}(\mathcal{O}_F[[X_1, \dots, X_n]])$ is a closed affine subscheme that after a multiplicative change of variables is a finite union of intersections of hyperplanes of the form $X_i = \zeta_i - 1$ for $\zeta_i \in \mu_{p^\infty}$.

We can now state the p -adic generalization as follows:

Theorem 2.5. *Let $A = \mathcal{O}_F[[X_1, \dots, X_n]]/I$. An irreducible component of the Zariski closure of the special points \mathcal{S}_A on $\text{Spec}(A)$ is a special multiplicative subscheme.*

We also establish the stronger result in this p -adic setting:

Theorem 2.6. *Let $A = \mathcal{O}_F[[X_1, \dots, X_n]]/I$. Then exactly one of the following occurs:*

- (1) *The affine scheme $\text{Spec}(A)$ contains a positive dimensional special multiplicative subscheme.*
- (2) *There exist an explicit constant $C_I > 0$, depending only on I and the choice of p -adic absolute value $|\cdot|_p$ on $\mathbb{B}^n(0, \mathbb{C}_p)$, as well as a finite set $\mathcal{F} \subset \mathcal{S}^n$ such that for any $\phi \in I$,*

$$|\phi(\zeta_1 - 1, \dots, \zeta_n - 1)|_p > C_\phi$$

$$\text{if } (\zeta_1 - 1, \dots, \zeta_n - 1) \in \mathcal{S}^n \setminus \mathcal{F}.$$

2.3. Almost vanishing loci. Let π denote a uniformizer for \mathcal{O}_F and $\mathbb{F}_q = \mathcal{O}_F/\pi\mathcal{O}_F$ the residue field. For any $\epsilon > 0$ and any ideal $I \subset \mathcal{O}_F[[X_1, \dots, X_n]]$, we consider the special points lying close to $\text{Spec}(\mathcal{O}_F[[X_1, \dots, X_n]]/I)$:

$$S_I(\epsilon) := \{(\zeta_1 - 1, \dots, \zeta_n - 1) \in \Omega^n \text{ such that } \forall \phi \in I, |\phi(\zeta_1 - 1, \dots, \zeta_n - 1)|_p < \epsilon\}.$$

We note that $\mathcal{O}_F[[X_1, \dots, X_n]]$ is a Noetherian ring and that it suffices to check the conditions on the finitely many generators of I . If $I = (\phi)$, we simply write $S_\phi(\epsilon)$.

The endomorphisms of $\widehat{\mathbb{G}}_m^n$ transform $S_I(\epsilon)$. In particular, performing a multiplicative change of variables on $\mathcal{O}_F[[X_1, \dots, X_n]]/I$ acts on $S_I(\epsilon)$ via an automorphism of $\widehat{\mathbb{G}}_m^n$ as follows:

$$(2.1) \quad \begin{pmatrix} 1 & 0 & \dots & 0 \\ B_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \dots & 1 \end{pmatrix} : \begin{pmatrix} \zeta_1 - 1 \\ \zeta_2 - 1 \\ \vdots \\ \zeta_n - 1 \end{pmatrix} \mapsto \begin{pmatrix} \zeta_1 - 1 \\ \zeta_2 \zeta_1^{B_{21}} - 1 \\ \vdots \\ \zeta_n \prod_{1 \leq i < n} \zeta_i^{B_{ni}} - 1 \end{pmatrix}$$

Since our goal is to understand when $S_I(\epsilon)$ can be infinite, we may after twisting by an automorphism arrange for an explicit subsequence of special points:

Lemma 2.7. *Assume that $S_I(\epsilon)$ is infinite for some $\epsilon > 0$. Then after a multiplicative change of variables there is a sequence $\{\zeta_k\}_{k \in \mathbb{N}} \in \mu_{p^\infty}$ for which*

$$(\zeta_k - 1, \zeta_k^{a_{2k}} - 1, \zeta_k^{a_{2k}a_{3k}} - 1, \dots, \zeta_k^{a_{2k}a_{3k}\dots a_{nk}} - 1) \in S_I(\epsilon)$$

and:

- *The set $\{\zeta_k \in \mu_{p^\infty} | k \in \mathbb{N}\}$ is infinite.*
- *The set $\{\zeta_k^{a_{2k}a_{3k}\dots a_{nk}} | k \in \mathbb{N}\}$ is finite or each sequence $\{a_{ik}\}_{k \in \mathbb{N}} \in \mathbb{Z}_p$ converges to zero p -adically.*

Proof. We may swap the roles of the indeterminates X_i so that infinitely often elements $(\zeta_1 - 1, \dots, \zeta_n - 1) \in S_I(\epsilon)$ have decreasing orders, or equivalently $v_p(\zeta_i - 1) \leq v_p(\zeta_{i+1} - 1)$. Therefore there is an infinite sequence $\{\zeta_k \in \mu_{p^\infty} | k \in \mathbb{N}\}$ with

$$(\zeta_k - 1, \zeta_k^{a_{2k}} - 1, \zeta_k^{a_{2k}a_{3k}} - 1, \dots, \zeta_k^{a_{2k}a_{3k}\dots a_{nk}} - 1) \in S_I(\epsilon),$$

for exponents $\{a_{ik}\}_{k \in \mathbb{N}} \in \mathbb{Z}_p$. By compactness of \mathbb{Z}_p , we may after passing to a subsequence assume that the exponents a_{ik} converge p -adically to $A_i \in \mathbb{Z}_p$. We may now find $B_{i,j} \in \mathbb{Z}_p$ with

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ B_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \dots & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ A_2 \\ \vdots \\ A_2 \cdots A_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and twist by the corresponding multiplicative change of variables. It follows from (2.1) that all the exponent sequences $a_{2k} \dots a_{jk}$ converge to zero, in particular $A_2 = 0$. Now apply the same procedure to the $(n-1)$ last coordinates

$$(\xi_k - 1, \xi_k^{a_{3k}} - 1, \dots, \xi_k^{a_{3k} \cdots a_{nk}} - 1)$$

where $\xi_k = \zeta_k^{a_{2k}}$ to arrange for $A_3 = 0$. Iterating this process, we get $A_i = 0$ for $2 \leq i \leq n$. \square

The non-invertible multiplication-by- p endomorphism of $\widehat{\mathbb{G}}_m^n$ on the other hand satisfies:

Lemma 2.8. *Assume $(\zeta_1 - 1, \dots, \zeta_n - 1) \in S_I(\epsilon)$ for some $\epsilon \leq |\pi|_p$. Then*

$$(\zeta_1^p - 1, \dots, \zeta_n^p - 1) \in S_I(\epsilon^p).$$

Proof. This follows from evaluating at roots of unity the congruence

$$\phi((X_1 + 1)^p - 1, \dots, (X_n + 1)^p - 1) \equiv \phi(X_1, \dots, X_n)^p \pmod{\pi}$$

for functions $\phi \in I$. \square

We turn to the proof of the general rigidity result for power series:

Proposition 2.9. *Let $\phi \in \mathcal{O}_F[[X_1, \dots, X_n]]$ be a power series. Then exactly one of the following occurs:*

- (1) *After a multiplicative change of variables, there are fixed $(\xi_2 - 1, \dots, \xi_n - 1) \in \mathcal{S}^{n-1}$ such that $\phi(s, \xi_2 - 1, \dots, \xi_n - 1) = 0$ for all $s \in \mathbb{B}(0, \mathbb{C}_p)$.*
- (2) *There exist a constant $C_\phi > 0$ and a finite set \mathcal{F}_ϕ such that for all $(\zeta_1 - 1, \dots, \zeta_n - 1) \in \mathcal{S}^n - \mathcal{F}_\phi$ there is a lower bound*

$$|\phi(\zeta_1 - 1, \dots, \zeta_n - 1)|_p \geq C_\phi.$$

The proof proceeds by induction and the following lemma deals with the base case $n = 1$.

Lemma 2.10. *Let $\phi \in \mathcal{O}_F[[X]]$ be a power series and $\{r_k\}_{k \in \mathbb{N}} \in \mathfrak{m}_{\mathbb{C}_p} = \mathbb{B}(0, \mathbb{C}_p)$ be a sequence so that*

$$|\phi(r_k)|_p < \epsilon$$

for some $0 \leq \epsilon < 1$. If $v_p(r_k) \rightarrow 0$ then $\phi \in \pi \mathcal{O}_F[[X]]$.

Proof. Assume $\phi \notin \pi \mathcal{O}_F[[X]]$. Writing $\phi = \sum_{n=0}^{\infty} a_n X^n$, let M be the smallest integer such that $v_p(a_M) = 0$. Then, provided k is large enough so that $v_p(r_k) < v_p(\pi)$, we have

$$v_p(\phi(r_k)) = v_p(a_M r_k^M) = M v_p(r_k)$$

However, the right hand side becomes arbitrarily small, which is a contradiction. \square

The next lemma is crucial and provides the induction step in our proof. It puts a strong restriction on the subsets of \mathcal{S}^n that can be realized as $S_\phi(\epsilon)$ for some $\phi \in \mathcal{O}_F[[X_1, \dots, X_n]]$.

Lemma 2.11. *For any power series $\phi \in \mathcal{O}_F[[X_1, \dots, X_n]]$, if*

$$(\zeta_k - 1, \zeta_k^{a_{2k}} - 1, \dots, \zeta_k^{a_{2k}a_{3k}\dots a_{nk}} - 1) \in S_\phi(p^{-(1-c)v_p(\pi)}),$$

where $0 \leq c < 1$ is a constant and each $\{a_{ik}\}_{k \in \mathbb{N}} \in \mathbb{Z}_p$ converges to zero for an infinite sequence $\{\zeta_k\}_{k \in \mathbb{N}} \in \mu_{p^\infty}$, then either:

- (1) the sequence $\{\zeta_k^{a_{2k}a_{3k}\dots a_{nk}}\}_{k \in \mathbb{N}}$ belongs to a finite set of roots of unity or
- (2) the power series $\phi \in \pi \mathcal{O}_F[[X_1, \dots, X_n]]$.

Proof. We proceed by induction. If $n = 1$ the result follows from Lemma 2.10. For arbitrary n , let c_k denote the order of $\zeta_k^{a_{2k}a_{3k}\dots a_{nk}}$. Assume the set $\{\zeta_k^{a_{2k}a_{3k}\dots a_{nk}} | k \in \mathbb{N}\}$ is infinite, then the same is true of $\{c_k | k \in \mathbb{N}\}$. We want to show $\phi \in \pi \mathcal{O}_F[[X_1, \dots, X_n]]$. Suppose not; there is then a largest integer M with X_n^M dividing the reduction $\bar{\phi} \in \mathbb{F}_q[[X_1, \dots, X_n]]$ modulo π . We choose $\psi \in \mathcal{O}_F[[X_1, \dots, X_n]]$ so that

$$\phi(X_1, \dots, X_n) \equiv X_n^M \psi(X_1, \dots, X_n) \pmod{\pi}.$$

Since $v_p(\zeta_k^{a_{2k}a_{3k}\dots a_{nk}} - 1)$ becomes arbitrarily small, after passing to a subsequence

$$(\zeta_k - 1, \zeta_k^{a_{2k}} - 1, \dots, \zeta_k^{a_{2k}a_{3k}\dots a_{nk}} - 1) \in S_\psi(p^{-(1-c/2)v_p(\pi)})$$

for infinitely many $\zeta_k \in \mu_{p^\infty}$. It follows from Lemma 2.8 that:

$$(\zeta_k, \zeta_k^{c_k a_{2k}}, \dots, \zeta_k^{c_k a_{2k} \dots a_{(n-1)k}}, 0) \in S_\psi(p^{-(1-c/2)v_p(\pi)}).$$

By definition of c_k the set $\{\zeta_k^{c_k a_{2k} a_{3k} \dots a_{(n-1)k}} | k \in \mathbb{N}\}$ consists of primitive $p^{v_p(a_{nk})}$ -roots of unity. Since $a_{nk} \neq 0$ infinitely often by assumption, the set is infinite. Moreover the exponent sequence $\{a'_{2k} := c_k \cdot a_{2k}\}_{k \in \mathbb{N}}$ still converges to zero. Therefore by induction

$$\psi(X_1, \dots, X_{n-1}, 0) \in \pi \mathcal{O}_F[[X_1, \dots, X_{n-1}]].$$

We may write $\psi(X_1, \dots, X_n) = \psi(X_1, \dots, X_{n-1}, 0) + X_n \theta(X_1, \dots, X_n)$ for some power series $\theta \in \mathcal{O}_F[[X_1, \dots, X_n]]$, so that $\psi \equiv X_n \theta \pmod{\pi}$. This results in the congruence

$$\phi(X_1, \dots, X_n) \equiv X_n^{M+1} \theta(X_1, \dots, X_n) \pmod{\pi},$$

which contradicts the maximality of M . \square

We now prove the main result of this section.

Proof of Proposition 2.9. We proceed by induction on n . Assume that for any integer $M \geq 1$ the set $S_\phi(p^{-Mv_p(\pi)})$ is infinite. If $n = 1$ it follows from the Weierstrass Preparation Theorem or Lemma 2.10 that ϕ vanishes identically. For $n > 1$, after applying Lemma 2.7 we conclude that there are infinitely many $\zeta_k \in \mu_{p^\infty}$ for which

$$(\zeta_k - 1, \zeta_k^{a_{2k}} - 1, \dots, \zeta_k^{a_{2k}a_{3k}\dots a_{nk}} - 1) \in S_\phi(p^{-Mv_p(\pi)})$$

for all M . By Lemma 2.11 the sequence $\{\zeta_k^{a_{2k}a_{3k}\dots a_{nk}}\}_{k \in \mathbb{N}}$ must belong to a finite set of roots of unity. Thus there is a fixed $\xi \in \mu_{p^\infty}$ such that

$$|\phi(X_1, \dots, X_{n-1}, \xi - 1)|_p$$

is arbitrarily small for infinitely many roots of unity in \mathcal{S}^{n-1} . We conclude by applying our induction hypothesis to $\phi(X_1, \dots, X_{n-1}, \xi - 1) \in \mathcal{O}_{F[\xi]}[[X_1, \dots, X_{n-1}]]$. \square

We obtain in the same way:

Proof of Theorem 2.6. The ideal $I \subset \mathcal{O}_F[[X_1, \dots, X_n]]$ is finitely generated. Writing $I = (\phi_1, \dots, \phi_d)$, we run the proof above for the d generators simultaneously. \square

When $n = 2$, we get the explicit formulation:

Proposition 2.12. *Let $\phi \in \mathcal{O}_F[[X, Y]]$ be an irreducible power series passing through the origin. If*

$$\phi(\zeta - 1, \xi - 1) = 0$$

for infinitely many pairs of p -power roots of unity (ζ, ξ) , then

$$\phi = (X + 1)^m - (Y + 1)$$

for some $m \in \mathbb{Z}_p$, after possibly switching the roles of X and Y .

Proof. After possibly switching the roles of X and Y , it follows from Lemmas 2.10 and 2.7 that there are fixed $m \in \mathbb{Z}_p$ and $\xi \in \mu_{p^\infty}$ such that $\phi(X, (Y + 1)(X + 1)^m - 1)$ vanishes at $(\zeta_k - 1, \xi - 1)$ for an infinite sequence $\{\zeta_k\}_{k \in \mathbb{N}}$. We deduce that

$$\phi(\zeta_k - 1, \xi \zeta_k^m - 1) = 0$$

for an infinite sequence $\{\zeta_k\}_{k \in \mathbb{N}}$. Over $\mathcal{O}_{F[\xi]}$ we may then write

$$\phi(X, Y) = \phi(X, \xi(X + 1)^m - 1) + (\xi(X + 1)^m - (Y + 1))G(X, Y)$$

for some power series $G(X, Y)$. Since the power series $H(X) := \phi(X, \xi(X + 1)^m - 1)$ vanishes at infinitely many points $\{\zeta_k - 1\}_{k \in \mathbb{N}}$ it follows that $H = 0$ and thus

$$(2.2) \quad \phi(X, Y) = (\xi(X + 1)^m - (Y + 1))G(X, Y)$$

for some $G(X, Y) \in \mathcal{O}_{F[\xi]}[[X, Y]]$. Taking conjugates under the group $\text{Gal}(F[\xi]/F)$ of order g in (2.2), it follows that

$$(2.3) \quad \phi(X, Y)^g = \left(\prod_{\sigma \in \text{Gal}(F[\xi]/F)} ((X + 1)^m - \sigma(\xi)(Y + 1)) \right) \left(\prod_{\sigma \in \text{Gal}(F[\xi]/F)} G^\sigma(X, Y) \right)$$

where $G^\sigma(X, Y)$ is obtained from G by acting on the coefficients. The two factors on the right hand side of (2.3) have coefficients in \mathcal{O}_F . Since ϕ is irreducible, it must be an irreducible factor of $\phi = \prod_{\sigma \in \text{Gal}(F[\xi]/F)} ((X + 1)^m - \sigma(\xi)(Y + 1))$. But requiring $\phi(0, 0) = 0$ forces $\xi = g = 1$ which shows that $\phi(X, Y) = (X + 1)^m - (Y + 1)$, as desired. \square

2.4. Geometric statements. We are interested in irreducible components Z of the Zariski closure of the set of special points \mathcal{S}_A on $A = \mathcal{O}_F[[X_1, \dots, X_n]]/I$ for I a non-trivial ideal. As irreducible special subschemes correspond to products of a formal subtorus with a torsion point of $\widehat{\mathbb{G}}_m^n$, their dimension should be the dimension of the subtorus. While one has to be careful with dimensions in our setting as the ring of coefficients is one-dimensional, it follows from the definitions of Z that $(\pi) \subset \mathcal{O}_F$ is never an associated prime. The codimension of Z can be read off on the set of special points:

Lemma 2.13. *Let Z be an irreducible component of the Zariski closure of \mathcal{S}_A and let S_Z denote the set of special points lying on Z . Then Z has codimension the maximum number r of columns with finite projections in the $\text{Aut}(\widehat{\mathbb{G}}_m^n) \cong \text{GL}_n(\mathbb{Z}_p)$ -orbit of S_Z .*

Proof. The inequality $\text{codim}(Z) \geq r$ is straightforward: if r coordinates have finite projections we may cover S_Z by finitely many sets of the form $\mathcal{S}^r \times (\xi_1 - 1) \times \dots \times (\xi_{n-r} - 1)$. Their Zariski closure are codimension r hyperplanes. We proceed to show $\text{codim}(Z) \leq r$. We may

assume $\text{codim}(Z) \geq 1$ and $n > r$. By definition of r , after a multiplicative change of variables as in Lemma 2.7 we may assume there are sequences $(\zeta_k)_{k \in \mathbb{N}}$ and $(a_{ik})_{k \in \mathbb{N}}$ with

$$S_0 := \{(\zeta_k - 1, \zeta_k^{a_{2k}} - 1, \zeta_k^{a_{2k}a_{3k}} - 1, \dots, \zeta_k^{a_{2k}a_{3k}\dots a_{nk}} - 1) | k \in \mathbb{N}\} \subseteq S_Z$$

and the projection $\{\zeta_k^{a_{2k}\dots a_{(n-r)k}} - 1 | k \in \mathbb{N}\}$ to coordinate $(n-r)$ of S_0 is an infinite set. Let $r' \leq r$ be such that coordinate $(n-r')$ of S_0 is the last one with infinite projection. In particular, there is a height r' prime ideal $\mathfrak{p}_0 = (X_{n-r'+1} - \xi_{n-r'+1}, \dots, X_n - \xi_n)$ of Z for some fixed roots of unity $(\xi_{n-r'+1}, \dots, \xi_n) \in \mu_{p^\infty}^{r'}$. Suppose $\text{codim}(Z) > r$. Thus we may find a height $r+1$ prime ideal \mathfrak{p} strictly containing \mathfrak{p}_0 and pick $f \in \mathfrak{p}$ that is not in (π, \mathfrak{p}_0) . Then

$$f(X_1, \dots, X_{n-r'}, \xi_{n-r'+1} - 1, \dots, \xi_n - 1) \in \mathcal{O}_{F[\xi_{n-r'+1}, \dots, \xi_n]}[[X_1, \dots, X_{n-r'}]] \neq 0$$

and we may apply Lemma 2.11 to $f(X_1, \dots, X_{n-r'}, \xi_{n-r'+1} - 1, \dots, \xi_n - 1)$, whence the projection to coordinate $(n-r')$ of S_0 is a finite set, a contradiction. \square

We now prove the geometric formulation of our generalization of the multiplicative Manin-Mumford Conjecture:

Theorem 2.14. *Let $A = \mathcal{O}_F[[X_1, \dots, X_n]]/I$. An irreducible component of the Zariski closure of the special points \mathcal{S}_A on $\text{Spec}(A)$ is a special multiplicative subscheme.*

Proof. We denote by Z an irreducible component of the Zariski closure of \mathcal{S}_A and by S_Z the special points on Z . We apply a multiplicative change of variables realizing $r = \text{codim}(Z)$ as the number of columns of the set S_Z with finite projection by Lemma 2.13. There are then finitely many sets $H_{r,i} = \mathcal{S}^{n-r} \times (\xi_{i,n-r+1} - 1) \times \dots \times (\xi_{i,n} - 1)$ with

$$S_Z \subset \bigcup_{i=1}^f H_{r,i}.$$

Therefore by definition $Z \subset \bigcup_{i=1}^f Z_{r,i}$, where $Z_{r,i}$ denotes the Zariski closure of $H_{r,i}$. Since the $Z_{r,i}$ are irreducible special multiplicative subschemes of the same codimension as Z , it must be that $Z = Z_{r,i}$ for some i , as desired. \square

In particular, we note that:

Corollary 2.15. *With notations as above, if the Zariski closure of \mathcal{S}_A is d -dimensional, then $\text{Spec}(A)$ contains a d -dimensional special multiplicative subscheme.*

One also deduces the rigidity result for formal schemes:

Corollary 2.16. *If the closed formal subscheme $\text{Spf}(A) \subset \widehat{\mathbb{G}}_m^n$ contains infinitely many $\overline{\mathbb{Q}}_p$ -torsion points of $\widehat{\mathbb{G}}_m^n$, it contains a translate of a formal subtorus by a torsion point. Moreover, if $\text{Spf}(A)$ is geometrically irreducible and the set of special points \mathcal{S}_A are Zariski dense, $\text{Spf}(A)$ is precisely the translate of a formal subtorus by a torsion point.*

Proof. The formal subschemes associated to special subschemes are translates of formal subtori. Therefore the statement follows from Theorem 2.14. \square

3. FURTHER GENERALIZATIONS

At this point, the reader may wonder whether similar results hold for a larger class of formal groups than the multiplicative group. In general, we may consider an n -dimensional formal Lie group \mathcal{F} over a complete Noetherian local ring R with residue field of characteristic p and the corresponding connected p -divisible group $\mathcal{F}[p^\infty]$. The special points are then torsion points in $\mathcal{F}[p^\infty](\overline{K})$, where K is the field of fractions of R . One may ask:

Question 3.1. *Is there a class of special subschemes of $\text{Spec}(R[[X_1, \dots, X_n]])$ such that the analogue of Theorem 2.14 holds?*

We proceed to show that essentially all the results of Section 2 hold replacing $\widehat{\mathbb{G}}_m$ by a one-dimensional Lubin-Tate formal group \mathcal{F}_{LT} . The case of replacing $\widehat{\mathbb{G}}_m^n$ by a generalized n -dimensional Lubin-Tate formal group or the formal Lie group of an abelian scheme will be addressed in future work.

3.1. Products of one-dimensional Lubin-Tate formal groups. Let E/\mathbb{Q}_p be a subfield of F with ring of integers \mathcal{O}_E , uniformizer π_E and residue field $\mathbb{F}_{q'}$. Given a power series $f \in \mathcal{O}_E[[X]]$ with

$$f(X) \equiv \pi_E X \pmod{X^2} \text{ and } f(X) \equiv X^{q'} \pmod{\pi_E},$$

Lubin and Tate show [LT65, Lemma 1] that for all $a \in \mathcal{O}_E$ there is a unique power series $[a](X) \in \mathcal{O}_E[[X]]$ with $[a](X) \equiv aX \pmod{X^2}$ and $f([a](X)) = [a](f(X))$. They construct [LT65, Theorem 1] a commutative one-dimensional formal group law $L(X, Y) \in \mathcal{O}_E[[X, Y]]$ such that for all $a, b \in \mathcal{O}_E$:

- (1) $L([a](X), [a](Y)) = [a](L(X, Y))$
- (2) $L([a](X), [b](X)) = [a + b](X)$
- (3) $[a]([b](X)) = [ab](X)$
- (4) $[\pi_E](X) = f(X)$ and $[1](X) = X$

Up to isomorphism, the group law is independent of the choice of $f \in \mathcal{O}_E[[X]]$ with the desired properties. We denote by \mathcal{F}_{LT} the formal group over \mathcal{O}_E resulting from this construction. In particular, the properties above show there is an injective ring homomorphism

$$\mathcal{O}_E \hookrightarrow \text{End}(\mathcal{F}_{LT}),$$

and the $\overline{\mathbb{Q}}_p$ -torsion points of \mathcal{F}_{LT} form a divisible \mathcal{O}_E -module. We will abuse notations and write $\mathcal{F}_{LT}[\pi_E^\infty]$ for the $\overline{\mathbb{Q}}_p$ -points. It follows from the Newton polygon of $[\pi_E](X)$ and the congruence $[\pi_E](X) \equiv X^{q'} \pmod{\pi_E}$ that a torsion point $\zeta \in \mathcal{F}_{LT}[\pi_E^\infty]$ of exact order π^k has normalized valuation $v_p(\zeta) = 1/(q^k - q^{k-1})$. Adjoining torsion points gives a totally ramified abelian extension $E(\mathcal{F}_{LT}[\pi_E^\infty])$ of E . As these properties suggest, we may take as set of *special points* the $\overline{\mathbb{Q}}_p$ -points in $\mathcal{F}_{LT}^n[\pi_E^\infty]$ and define as before:

Definition 3.2.

- (1) An \mathcal{F}_{LT} -multiplicative change of variables on $\mathcal{O}_F[[X_1, \dots, X_n]]$ is given by a series of transformations using the formal group law of \mathcal{F}_{LT} :

$$X_i \mapsto L(\cdots L(L(X_i, [B_{i-1}](X_{i-1})), [B_{i-2}](X_{i-2})), \dots, [B_1](X_1))$$

for $1 \leq i \leq n$ and $B_j \in \mathcal{O}_E$, composed with possibly swapping variables $X_i \leftrightarrow X_j$.

- (2) An \mathcal{F}_{LT} -special subscheme of $\text{Spec}(\mathcal{O}_F[[X_1, \dots, X_n]])$ is a closed subscheme that after an \mathcal{F}_{LT} -multiplicative change of variables becomes a finite union of intersections of hyperplanes $X_i = \zeta_i$ where $\zeta_i \in \mathcal{F}_{LT}[\pi^\infty]$.

Remark 3.3. The \mathcal{F}_{LT} -multiplicative changes of variables correspond to automorphisms of the Lubin-Tate formal group law. An irreducible scheme $\text{Spec}(\mathcal{O}_F[[X_1, \dots, X_n]]/I)$ is \mathcal{F}_{LT} -special if and only if $\text{Spf}(\mathcal{O}_F[[X_1, \dots, X_n]]/I)$ is the translate of a formal subtorus \mathcal{F}_{LT}^d by a torsion point of \mathcal{F}_{LT}^n . When $\mathcal{F}_{LT} = \widehat{\mathbb{G}}_m$, we have

$$L(X, Y) = (X + 1)(Y + 1) - 1$$

and we recover all the previous definitions as a special case.

We revisit the proofs of the key Lemmas 2.7 and 2.11 in this more general setting. Fix an ideal $I \subset \mathcal{O}_F[[X_1, \dots, X_n]]$ and let $A = \mathcal{O}_E[[X_1, \dots, X_n]]/I$. For any $\epsilon > 0$ we consider as before the special points almost on $\text{Spec}(A)$:

$$S_I(\epsilon) = \{(\zeta_1, \dots, \zeta_n) \in \mathcal{F}_{LT}^n[\pi^\infty] \text{ such that } \forall \phi \in I \ |\phi(\zeta_1, \dots, \zeta_n)|_p < \epsilon\}.$$

Using the properties of the endomorphism ring of \mathcal{F}_{LT}^n we again obtain the following results:

Lemma 3.4. *Assume that $S_I(\epsilon)$ is infinite for some $\epsilon > 0$. Then after an \mathcal{F}_{LT} -multiplicative change of variables there is a sequence $\{\zeta_k\}_{k \in \mathbb{N}} \in \mathcal{F}_{LT}[\pi^\infty]$ for which*

$$(\zeta_k, [a_{2k}](\zeta_k), \dots, [a_{2k} \cdots a_{nk}](\zeta_k)) \in S_I(\epsilon)$$

and:

- The set $\{\zeta_k \in \mathcal{F}_{LT}[\pi^\infty] | k \in \mathbb{N}\}$ is infinite.
- The set $\{[a_{2k} \cdots a_{nk}](\zeta_k) | k \in \mathbb{N}\}$ is finite or each sequence $\{a_{ik}\}_{k \in \mathbb{N}} \in \mathcal{O}_E$ converges to zero p -adically.

Proof. After swapping indeterminates X_i so that infinitely often elements $(\zeta_1, \dots, \zeta_n) \in S_I(\epsilon)$ have decreasing orders, we again get an explicit sequence

$$(\zeta_k, [a_{2k}](\zeta_k), \dots, [a_{2k} \cdots a_{nk}](\zeta_k)) \in S_I(\epsilon),$$

for exponents $a_{ik} \in \mathcal{O}_E$ and infinitely many $\zeta_k \in \mathcal{F}_{LT}[\pi^\infty]$. By compactness of \mathcal{O}_E , we may after passing to a subsequence assume that the exponents a_{ik} converge p -adically to $A_i \in \mathcal{O}_E$. As before, choose $B_{i,j} \in \mathcal{O}_E$ such that

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ B_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ A_1 \\ \vdots \\ A_1 \cdots A_{n-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Using crucially that the map $\mathcal{O}_F \rightarrow \text{End}(\mathcal{F}_{LT})$ is a ring homomorphism, we see that the \mathcal{F}_{LT} -multiplicative change of variables corresponding to $B_{i,j}$ acts on our sequence via:

$$\begin{aligned} \begin{pmatrix} \zeta \\ [a_2](\zeta) \\ \vdots \\ [a_2 \cdots a_n](\zeta) \end{pmatrix} &\mapsto \begin{pmatrix} \zeta \\ L([B_{21}](\zeta), [a_2](\zeta)) \\ \vdots \\ L([B_{n1}](\zeta), L(\cdots L([B_{nn-1}][a_2 \cdots a_{n-1}](\zeta), [a_2 \cdots a_n](\zeta)) \cdots) \end{pmatrix} \\ &= \begin{pmatrix} \zeta \\ [B_{21} + a_2](\zeta) \\ \vdots \\ [B_{n1} + \cdots + B_{nn-1}a_2 \cdots a_{n-1} + a_2 \cdots a_n](\zeta) \end{pmatrix} \end{aligned}$$

so that after changing variables we may assume the products $a_{2k} \cdots a_{jk} \in \mathcal{O}_E$ converge to zero for any $2 \leq j \leq n$. Now repeat this process as in the proof of Lemma 2.7 to conclude. \square

Lemma 3.5. *Assume $(\zeta_1, \dots, \zeta_n) \in S_I(\epsilon)$ for some $\epsilon \leq |\pi|_p$. Then*

$$([\pi_E](\zeta_1), \dots, [\pi_E](\zeta_n)) \in S_I(\epsilon^{q'}).$$

Proof. The congruence $[\pi_E](X) \equiv X^{q'} \pmod{\pi_E}$ yields

$$\phi([\pi_E](X_1), \dots, [\pi_E](X_n)) \equiv \phi(X_1^{q'}, \dots, X_n^{q'}) \equiv \phi(X_1, \dots, X_n)^{q'} \pmod{\pi}$$

for functions $\phi \in I$ and the result follows. \square

It follows that the key Lemma 2.11 generalizes:

Lemma 3.6. *For any power series $\phi \in \mathcal{O}_F[[X_1, \dots, X_n]]$, if*

$$(\zeta_k, [a_{2k}](\zeta_k), \dots, [a_{2k} \cdots a_{nk}](\zeta_k)) \in S_\phi(p^{-(1-c)v_p(\pi)}),$$

where $0 \leq c < 1$ is a constant and each sequence $\{a_{ik}\}_{k \in \mathbb{N}} \in \mathcal{O}_E$ converges to zero for an infinite set $\{\zeta_k | k \in \mathbb{N}\} \subset \mathcal{F}_{LT}[\pi_E^\infty]$, then either:

- (1) *the sequence $\{[a_{2k}a_{3k} \cdots a_{nk}](\zeta_k)\}_{k \in \mathbb{N}}$ belongs to a finite set of torsion points or*
- (2) *the power series $\phi \in \pi \mathcal{O}_F[[X_1, \dots, X_n]]$.*

Proof. We proceed by induction. If $n = 1$ the result again follows from Lemma 2.10. For arbitrary n , let c_k be the smallest power π_E^k such that $[\pi_E^k] \cdot ([a_{2k} \cdots a_{nk}](\zeta_k)) = 0$. Assume the set $\{[a_{2k} \cdots a_{nk}](\zeta_k) | k \in \mathbb{N}\}$ is infinite, then the same is true of $\{c_k | k \in \mathbb{N}\}$. We want to show $\phi \in \pi \mathcal{O}_F[[X_1, \dots, X_n]]$. Suppose not; there is then a largest integer M with X_n^M dividing the reduction $\bar{\phi} \in \mathbb{F}_q[[X_1, \dots, X_n]]$ modulo π . We choose $\psi \in \mathcal{O}_F[[X_1, \dots, X_n]]$ so that

$$\phi(X_1, \dots, X_n) \equiv X_n^M \psi(X_1, \dots, X_n) \pmod{\pi}.$$

Since the valuation $v_p([a_{2k} \cdots a_{nk}](\zeta_k))$ becomes arbitrarily small, after passing to a subsequence we get

$$(\zeta_k, [a_{2k}](\zeta_k), \dots, [a_{2k} \cdots a_{nk}](\zeta_k)) \in S_\psi(p^{-(1-c/2)v_p(\pi)}).$$

We now see from Lemma 3.5 that:

$$(\zeta_k, [c_k a_{2k}](\zeta_k), \dots, [c_k a_{2k} \cdots a_{(n-1)k}](\zeta_k), 0) \in S_\psi(p^{-(1-c/2)v_p(\pi)}).$$

By definition of c_k the torsion point $[c_k a_{2k} \cdots a_{(n-1)k}](\zeta_k)$ has exact order the largest power of π dividing $[a_{nk}]$. Since $a_{nk} \neq 0$ infinitely often by assumption, $\{[c_k a_{2k} \cdots a_{(n-1)k}](\zeta_k) | k \in \mathbb{N}\}$

is infinite. It follows by induction that $\psi(X_1, \dots, X_{n-1}, 0) \in \pi \mathcal{O}_F[[X_1, \dots, X_{n-1}]]$. We may now write $\psi(X_1, \dots, X_n) = \psi(X_1, \dots, X_{n-1}, 0) + X_n \theta(X_1, \dots, X_n)$ for some power series $\theta \in \mathcal{O}_F[[X_1, \dots, X_n]]$, so that $\psi \equiv X_n \theta \pmod{\pi}$. This results in the congruence

$$\phi(X_1, \dots, X_n) \equiv X_n^{M+1} \theta(X_1, \dots, X_n) \pmod{\pi},$$

which contradicts the maximality of M . \square

The main results are now deduced in exactly the same way as in Section 2 for $\mathcal{F}_{LT} = \widehat{\mathbb{G}}_m$. Below are the adapted statements of the geometric formulations. The proof is left to the reader.

Theorem 3.7. *Let $A = \mathcal{O}_F[[X_1, \dots, X_n]]/I$. An irreducible component of the Zariski closure of the \mathcal{F}_{LT} -special points on $\text{Spec}(A)$ is a special \mathcal{F}_{LT} -multiplicative subscheme.*

Theorem 3.8. *Let $A = \mathcal{O}_F[[X_1, \dots, X_n]]/I$. Then exactly one of the following occurs:*

- (1) *The formal scheme $\text{Spf}(A)$ contains a translate of the Lubin-Tate formal group \mathcal{F}_{LT} by a torsion point of \mathcal{F}_{LT}^n .*
- (2) *There exist an explicit constant $C_I > 0$, depending only on I and the choice of p -adic absolute value $|\cdot|_p$, as well as a finite set $\mathcal{F} \subset \mathcal{F}_{LT}[\pi_E^\infty]^n$ such that for any $\phi \in I$,*

$$|\phi(\zeta_1, \dots, \zeta_n)|_p > C_I$$

if $(\zeta_1, \dots, \zeta_n) \in \mathcal{F}_{LT}[\pi_E^\infty]^n \setminus \mathcal{F}$.

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REFERENCES

- [Cha05] Ching-Li Chai. Hecke orbits on Siegel modular varieties. In *Geometric methods in algebra and number theory*, volume 235 of *Progr. Math.*, pages 71–107. Birkhäuser Boston, Boston, MA, 2005.
- [Cha08] Ching-Li Chai. A rigidity result for p -divisible formal groups. *Asian J. Math.*, 12(2):193–202, 2008.
- [Hid10] Haruzo Hida. The Iwasawa μ -invariant of p -adic Hecke L -functions. *Ann. of Math. (2)*, 172(1):41–137, 2010.
- [Hid11] Haruzo Hida. Constancy of adjoint \mathcal{L} -invariant. *J. Number Theory*, 131(7):1331–1346, 2011.
- [Hid14] Haruzo Hida. Hecke fields of Hilbert modular analytic families. In *Automorphic forms and related geometry: assessing the legacy of I. I. Piatetski-Shapiro*, volume 614 of *Contemp. Math.*, pages 97–137. Amer. Math. Soc., Providence, RI, 2014.
- [Lan60] Serge Lang. Integral points on curves. *Inst. Hautes Études Sci. Publ. Math.*, (6):27–43, 1960.
- [Lan65] Serge Lang. Division points on curves. *Ann. Mat. Pura Appl. (4)*, 70:229–234, 1965.
- [LT65] Jonathan Lubin and John Tate. Formal complex multiplication in local fields. *Ann. of Math. (2)*, 81:380–387, 1965.
- [Mon81] Paul Monsky. On p -adic power series. *Math. Ann.*, 255(2):217–227, 1981.
- [Nei02] Ana Raissa Berardo Neira. *Power series in p -adic roots of unity*. ProQuest LLC, Ann Arbor, MI, 2002. Thesis (Ph.D.)—The University of Texas at Austin.
- [Ray83] M. Raynaud. Courbes sur une variété abélienne et points de torsion. *Invent. Math.*, 71(1):207–233, 1983.
- [Sca98] Thomas Scanlon. p -adic distance from torsion points of semi-abelian varieties. *J. Reine Angew. Math.*, 499:225–236, 1998.

- [TV96] John Tate and José Felipe Voloch. Linear forms in p -adic roots of unity. *Internat. Math. Res. Notices*, (12):589–601, 1996.
- [Ull98] Emmanuel Ullmo. Positivité et discrétion des points algébriques des courbes. *Ann. of Math. (2)*, 147(1):167–179, 1998.
- [Ull07] Emmanuel Ullmo. Manin-Mumford, André-Oort, the equidistribution point of view. In *Equidistribution in number theory, an introduction*, volume 237 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 103–138. Springer, Dordrecht, 2007.
- [Zha95] Shou-Wu Zhang. Positive line bundles on arithmetic varieties. *J. Amer. Math. Soc.*, 8(1):187–221, 1995.
- [Zha98] Shou-Wu Zhang. Equidistribution of small points on abelian varieties. *Ann. of Math. (2)*, 147(1):159–165, 1998.

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